Recent advances on noncommutative differentially subordinate martingales theory

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The 15th Workshop on Markov Processes and Related Topics
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Joint with A. Osękowski, N. Randrianantoanina, L. Wu, and D. Zhou

Overview

Classical Differential Subordinations

Backgrounds

- The classical differential subordination of martingales was introduced by Burkholder in the eighties.

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- The classical differential subordination of martingales was introduced by Burkholder in the eighties.
- Let $x = (x_n)_n$, $y = (y_n)_n$ be two martingales and let $dx = (dx_n)_n$ and $dy = (dy_n)_n$ be the martingale difference sequences associated with them. Then y is said to be **differentially subordinate** to x if for any n, we have

$$|dy_n| \leq |dx_n|.$$

D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), 647–702.

Examples

- Martingale transforms: For instance, suppose that $x = (x_n)_{n\geq 0}$ is an arbitrary martingale, $v = (v_n)_{n\geq 0}$ is a predictable sequence with $|v_n| \leq 1$ and let y be the martingale transform of x by v, i.e.,

$$y_n = \sum_{k=0}^{n} v_k dx_k, \qquad n = 0, 1, 2, \ldots.$$

Then y is a martingale and we have $|dy_n| \leq |dx_n|$ for all n, so the differential subordination holds.

Examples

- Square functions: Let $x = (x_n)_{n\geq 0}$ be an arbitrary martingale taking values in some Hilbert space $\mathcal{H}$. Then we may treat x as a process taking values in a larger Hilbert space $\ell_2(\mathcal{H})$, simply embedding in onto the first coordinate, i.e.,

$$x \in \mathcal{H} \longrightarrow (x, 0, 0, \ldots) \in \ell_2(\mathcal{H}).$$

Consider another $\ell_2(\mathcal{H})$-valued martingale given by

$$y_n = (dx_0, dx_1, dx_2, \ldots, dx_n, 0, 0, \ldots).$$

Obviously,

$$|dy_n|_{\ell_2(\mathcal{H})} = |dx_n|_{\ell_2(\mathcal{H})}, \quad \forall n \geq 0.$$

On the other hand,

$$|y_n|_{\ell_2(\mathcal{H})} = \left(\sum_{k=0}^{n} |dx_k|^2 \right)^{1/2}.$$

Theorem (Burkholder, 1984)

Suppose that y is differentially subordinate to x. Then

$$\begin{aligned} &||y||_{1,\infty} \leq 2||x||_1; \\ &||y||_p \leq (p^* - 1)||x||_p, \quad 1 < p < \infty, \end{aligned} \tag{1}$$

where $p^* = \max\{p, p/(p-1)\}$. Here the constants are both sharp.

D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), 647–702.

- These estimates were extended in numerous directions. For example, continuous-time versions:

  G. Wang, *Differential subordination and strong differential subordination for continuous-time martingales and related sharp inequalities*, Ann. Probab. 23 (1995), no. 2. 522–551.

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- Moreover, such estimates have important applications in harmonic analysis and the theory of quasiconformal mappings.

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R. Bañuelos and G. Wang, *Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transformations, Duke Math. J. 80 (1995), 575-600.*

K. Astala, T. Iwaniec and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton University Press, 2009.

R. Bañuelos, A. Osękowski, *Sharp martingale inequalities and applications to Riesz transforms on manifolds, Lie groups and Gauss space.*, J. Funct. Anal. 269 (2015), no. 6, 1652–1713.

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Question

Can we define the notion of differential subordinations for noncommutative martingales?

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If so, can we extend Burkholder's results to the noncommutative case? Or in other words, can we obtain the noncommutative version:

$$\begin{aligned} &||y||_{1,\infty} \leq 2||x||_1; \\ &||y||_p \leq (p^* - 1)||x||_p, \quad 1 < p < \infty, \end{aligned} \tag{2}$$

where $p^* = \max\{p, p/(p-1)\}$.

Noncommutative Differential Subordinations

Noncommutative martingales

- $\mathcal{M}$ is a semifinite von Neumann algebra with a semifinite normal faithful trace τ.
- A noncommutative probability space $(\mathcal{M}, \tau)$: $\tau(1) = 1$.

Example 1. $\mathcal{M} = L_\infty(\Omega, P)$, $\tau = \int_\Omega$; $\quad \tau(1) = P(\Omega) = 1$
$(\mathcal{M}, \tau)$: the classical probability space

Example 2. $\mathcal{M} = \mathbb{M}_n(\mathbb{C})$, $\tau = \frac{1}{n} Tr$
$(\mathcal{M}, \tau)$: NC probability space

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- $L_0(\mathcal{M})$: the set of τ-measurable operators
- NC L_p-spaces: for $0 < p \leq \infty$,

$$L_p(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}$$

Noncommutative martingales

- $(\mathcal{M}_n)_n$ is an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ with $\bigcup_n \mathcal{M}_n$ is dense for the weak-operator topology in $\mathcal{M}$.
- $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ is a trace preserving conditional expectation.

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- A sequence $x = (x_n)_n$ in $L_1(\mathcal{M})$ is called a noncommutative martingale with respect to a filtration $(\mathcal{M}_n)_n$ if $\mathcal{E}_n(x_{n+1}) = x_n$.

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- L_p-bounded martingale: $1 \leq p \leq \infty$

$$\|x\|_p := \sup_n \|x_n\|_p < \infty$$

NC martingale theory

- G. Pisier and Q. Xu, *Non-commutative martingale inequalities*, Comm. Math. Phys. (1997)

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Main difficulties

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$$\Big\| \Big(\sum_n |x_n|^2 \Big)^{1/2} \Big\|_p \approx \Big\| \Big(\sum_n |x_n^*|^2 \Big)^{1/2} \|_p \quad ?$$

Answer: No!

Example. Let $(\mathcal{M}, \tau) = (M_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr})$. Set $x_k = e_{k,0}$, $0 \le k < n$. It is immediate that

$$\Big\| \Big(\sum_{k=0}^{n-1} |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} = n^{1/2-1/p}, \quad \Big\| \Big(\sum_{k=0}^{n-1} |x_k^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} = 1.$$

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Noncommutative Differential Subordinations

How to give an appropriate definition?

Classical definition

Let $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$ be two martingales, then x is differentially subordinate to y if

$$|dx_n| \leq |dy_n|, \quad \forall n \geq 0;$$

or equivalently,

$$|dx_n|^2 \leq |dy_n|^2, \quad \forall n \geq 0.$$

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The NC case?

How to give an appropriate definition?

Classical definition

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The NC case? NO!

Let $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$ be two NC self-adjoint martingales, then x is differentially subordinate to y if

$$|dx_n|^2 \leq |dy_n|^2, \quad \forall n \geq 0.$$

Example

Fix an even integer N and let ε_0, ε_1, ..., ε_N be independent Rademacher variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for each $n \geq 0$, $\mathcal{F}_n$ is the σ-algebra generated by ε_0, ε_1, ..., ε_n (with the convention $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \mathcal{F}$ if $n > N$). Consider

$$\mathcal{M} = L_\infty(\Omega, \mathcal{F}, \mathbb{P}) \bar{\otimes} \mathbb{M}_{N+2}$$

and the filtration

$$\mathcal{M}_n = L_\infty(\Omega, \mathcal{F}_n, \mathbb{P}) \bar{\otimes} \mathbb{M}_{N+2}, n = 0,\ 1,\ 2,\ \ldots .$$

Finally, consider the sequences $dx = (dx_n)_{n \geq 0}$, $dy = (dy_n)_{n \geq 0}$ given by $dx_n = \varepsilon_n \otimes (e_{1,n+2} + e_{n+2,1})$ and $dy_n = \varepsilon_n \otimes (e_{1,1} + e_{n+2,n+2})$, $n = 0, 1,\ 2,\ \ldots,\ N$; for remaining n, set $dx_n = dy_n = 0$. Then one may verify that

$$|dy_n|^2 \leq |dx_n|^2, \quad \forall n \geq 0.$$

However,

$$\tau(|y| \geq 1)/\tau(|x|) = \frac{N+2}{2\sqrt{N+1}} \longrightarrow \infty,$$

which means **the weak-type (1,1) estimate fails**.
Moreover, we have

$$||y||_p \geq (N+2)^{1/p}, \quad ||x||_p = 2^{1/p}\sqrt{N+1}.$$

Consequently, **the strong-type (p,p) estimate for $1 < p < 2$ also fails**.

Let x, y be two self-adjoint noncommutative martingales.

Definition (Osękowski, 2008, Probab. Theory Related Fields)

We say that y is *differentially subordinate* to x if
(i) for any $n \geq 0$ and any projection $R \in \mathcal{M}_n$ we have

$$\tau(Rdy_nRdy_nR) \leq \tau(Rdx_nRdx_nR).$$

(ii) for any $n \geq 0$ and any projections R, $S \in \mathcal{M}_n$ such that $R \wedge S = 0$ and $R + S \in \mathcal{M}_{n-1}$, we have

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Remark

- The weak type $(1,1)$-inequality is true under this domination.

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Remark

- The weak type $(1,1)$-inequality is true under this domination.
- $x, y \in L_2(\mathcal{M})$
- The second problem is the complexity of (ii), which makes it very difficult to be applied.

Definition (Jiao-Osękowski-Wu, Adv. Math., 2018)

We say that y is *weakly differentially subordinate* to x, if for any $n \geq 0$ and any projection $R \in \mathcal{M}_{n-1}$, we have

$$Rdy_nRdy_nR \leq Rdx_nRdx_nR. \tag{3}$$

We say that y is *very weakly differentially subordinate* to x if

$$dy_n^2 \leq dx_n^2 \qquad \text{for any } n \geq 0. \tag{4}$$

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Remark

- In the commutative case, the three definitions are identical!
- However, in the NC case
  differential subordination $\Longrightarrow$ weak differential subordination $\Longrightarrow$ very weak differential subordination

Main Theorems: Weak type inequality

Let x, y be two self-adjoint martingales.

Theorem 1 (Jiao-Osękowski-Wu, Adv. Math., 2018)

Suppose y is weakly differentially subordinate to x. Then for any $\lambda > 0$ we have

$$\lambda\tau(|y| \geq \lambda) \leq 36||x||_1, \quad x \in L_1(\mathcal{M}).$$

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Theorem 1' (Jiao-Randrianantoanina-Wu-Zhou, CMP, 2019)

Suppose y is weakly differentially subordinate to x. Then for any $\lambda > 0$ we have

$$\lambda\tau(|y| \geq \lambda) \leq C||x||_1, \quad x \in L_1(\mathcal{M}),$$

where $c = 2 + 2B^2 + \dfrac{4B}{B-1}$ for $B > 1$.

Main Theorems: Strong type inequalities

Theorem 2 (Jiao-Osękowski-Wu, Adv. Math., 2018)

Suppose that y is very weakly differentially subordinate to x. Then

$$||y||_p \leq c_p||x||_p, \quad 2 \leq p < \infty,$$

where $c_2 = 1$ and, for $p > 2$, $c_p = \frac{2^{1+1/p}p(1+2^{2-4/p})^{1/2}B^{(p+2)/2}}{(1-B^{2-p})^{1/2}}$ and $B = 1 + 1/p$.

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Remark

The constant in Theorem 2 is of order $O(p)$ as $p \to \infty$. It is already optimal in the commutative setting.

Main Theorems

Theorem 3 (Jiao-Osękowski-Wu, Adv. Math., 2018)

Suppose that y is weakly differentially subordinate to x. Then

$$\|y\|_p \leq c_p||x||_p, \quad 1 < p < 2,$$

where $c_p = \frac{4 \cdot 2^{p-1}}{2^{p-1}-1} \left(9 \cdot 2^p - 3 + \frac{4 \cdot 2^p(2^p-1)}{1-2^{p-2}}\right)^{1/p}$.

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Remark

The constant in Theorem 3 is of order $O((p-1)^{-1})$ as $p \to 1_+$. It is already optimal in the commutative setting.

Idea of Proofs: weak type

A new Gundy type decomposition

Theorem (AIM,2018/ CMP,2019)

Let $x=(x_n)_{n\geq 1}$ be a self-adjoint L_1-bounded martingale and y is a self-adjoint martingale that is weakly differentially subordinate to x. For any $\lambda>0$, there exist four martingales a, b, g, and υ:

- $y=a+b+g+\upsilon$;
- the martingale a satisfies: $\|a\|_2^2 \leq 2\lambda\|x\|_1$;
- the martingale b satisfies: $\sum_{n\geq 1}\|db_n\|_1 \leq 4\|x\|_1$;
- g and υ are L_1-martingales with:

$$\max\Big\{\lambda\tau\Big(\bigvee_{n\geq 1}\mathrm{supp}|dg_n|\Big),\ \lambda\tau\Big(\bigvee_{n\geq 1}\mathrm{supp}\,|d\upsilon_n^*|\Big)\Big\}\leq \|x\|_1.$$

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Remark

$$da_n := q_{n-1}dy_nq_n - \mathcal{E}_{n-1}(q_{n-1}dy_nq_n).$$

Idea of Proofs: strong type

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  weak (1,1)+ strong (2,2)$\longrightarrow$ strong (p,p) for $1 < p < 2$ by interpolation$\longrightarrow$strong (p,p) for $2 < p < \infty$ by duality
- Fully new idea:
  "Summation method" for $1 < p < 2$
  Estimates similar to NC good-λ inequality for $2 \leq p < \infty$

Idea of Proofs: strong type

$1 < p < 2$: Summation method+NC Gundy's decomposition

The main steps can be summarized as follows:

- Fix $\lambda > 0$. Let $R^{\lambda}_{-1} = \mathbf{1}$ and, for any $n \geq 0$, define

$$R^{\lambda}_{n} = R^{\lambda}_{n-1}\chi_{(-\lambda,\lambda)}(R^{\lambda}_{n-1}x_n R^{\lambda}_{n-1}).$$

- Fix $B > 1$. For $n \geq 0$ and $k \in \mathbb{Z}$, we set

$$P^{B^k}_{n} := \bigwedge_{\ell \geq k} R^{B^\ell}_{n}.$$

- Consider the 'maximal operator'

$$a_n := \sum_{k\in\mathbb{Z}} B^k (P^{B^{k+1}}_{n} - P^{B^k}_{n}).$$

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$$(a_n = \sum_{k\in\mathbb{Z}} B^k 1_{\{B^k \leq \max_{0\leq m\leq n}|x_m| < B^{k+1}\}}.)$$

- Then we have the following key lemmas.

Lemma 1 (Jiao-Osękowski-Wu, Adv. Math., 2018)

Let $1 < p < \infty$. Then $\|a_N\|_p \leq \frac{B^{p-1}}{B^{p-1}-1}\|x_N\|_p$ for any $N \geq 0$.

Lemma 2 (Jiao-Osękowski-Wu, Adv. Math., 2018)

For any $N \geq 0$ and $k \in \mathbb{Z}$, we have

$$\tau\left(|y_N| \geq 4B^k\right) \leq B^{-2k}\tau\left(R_N^{B^k} x_N R_N^{B^k} x_N R_N^{B^k}\right) + 9B^{-k}\tau\left((I - R_N^{B^k})|x_N|\right). \quad (5)$$

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- Multiply both sides of the inequality (5) by B^{kp} and take the sum over all $k \in \mathbb{Z}$.
- K-functional of x_N w.r.t. the interpolation couple (L_1, L_2)

Idea of Proofs

$2 \le p < \infty$: NC good-λ inequality

- We use the projections built from the martingale y. Let $S_{-1} = R_{-1} = \mathbf{1}$ and define

$$R_n = R_{n-1}\chi_{(-1,1)}(R_{n-1}y_nR_{n-1}),$$
$$S_n = S_{n-1}\chi_{(-c,c)}(S_{n-1}y_nS_{n-1}).$$

- We get the following important estimate

Theorem (Jiao-Osękowski-Wu, Adv. Math., 2018)

$$\tau(\mathbf{1} - S_N) \leq 4(c-1)^{-2}\tau((\mathbf{1} - R_N)(x_N^2 + b)).$$

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Remark: The general good-λ inequality (Acta. Math. 1971): $\mathbb{E}Y^p \leq c_p \mathbb{E}X^p$ if for some $\beta, \delta, \alpha_{\beta,\delta} > 0$,

$$\mathbb{P}(Y \geq \beta\lambda) \leq (X > \delta\lambda) + \alpha_{\beta,\delta}(Y \geq \lambda),$$

which is a long-standing open problem in the NC setting. The essential feature is that the left- and the right-hand side involve different level sets of the dominated random variable. Such phenomenon occurs in Lemma 4 (in classical case, the above bound becomes

$$\mathbb{P}(\max_{0\leq n\leq N} |y_n| \geq c) \leq 4(c-1)^2\mathbb{E}(x_N^2 + b)1_{\{\max_{0\leq n\leq N}|y_n|\geq 1\}}).$$

Application to B-G inequality

- Consider an arbitrary martingale $x = (x_n)_{n\geq 0}$ with some filtration $(\mathcal{M}_n)_{n\geq 0}$.
- Consider the larger von Neumann algebra

$$\bar{\mathcal{M}} = \mathbb{M}_{N+2} \otimes \mathcal{M}$$

equipped with the tensor product trace and the filtration

$$\bar{\mathcal{M}}_n = \mathbb{M}_{N+2} \bar{\otimes} \mathcal{M}_n, n = 0, 1, 2, \ldots.$$

- Let $\bar{y} = (\bar{y}_n)_{n\geq 0}$, $\bar{x} = (\bar{x}_n)_{n\geq 0}$ be two martingales on $\bar{\mathcal{M}}$ with the difference sequences $d\bar{x} = (d\bar{x}_n)_{n\geq 0}$, $d\bar{y} = (d\bar{y}_n)_{n\geq 0}$ given by

$$d\bar{x}_n = (e_{11} + e_{n+2,n+2}) \otimes dx_n$$

and

$$d\bar{y}_n = (e_{1,n+2} + e_{n+2,1}) \otimes dx_n, \quad n = 0, 1, 2, \ldots, N;$$

for remaining n, set $d\bar{x}_n = d\bar{y}_n = 0$.

- It is obvious that $d\bar{x}$ and $d\bar{y}$ are martingale differences and

$$d\bar{x}_n^2 = d\bar{y}_n^2 = (e_{11} + e_{n+1,n+1}) \otimes dx_n^2$$

for all n.

- In other words, the martingales $\bar{x}$ and $\bar{y}$ are differentially subordinate to each other. Therefore,

$$c_p^{-1}||\bar{x}_N||_p \leq ||\bar{y}_N||_p \leq c_p||\bar{x}_N||_p, \qquad p \geq 2.$$

- Note that

$$|\bar{x}_N| = e_{11} \otimes |x_N| + \sum_{n=0}^{N} e_{n+2,n+2} \otimes |dx_n|,$$

which implies that

$$||\bar{x}_N||_p = \left(||x_N||_p^p + \sum_{n=0}^{N} ||dx_k||_p^p \right)^{1/p}.$$

- Thus,

$$||x_N||_p \leq ||\bar{x}_N||_p \leq (1 + 2^{1-2/p})||x_N||_p.$$

- Compute that

$$\bar{y}_N^2 = e_{11} \otimes S_N^2(x) + zz^*,$$

where $z = \sum_{n=0}^N e_{1,n+2} \otimes dx_n$.

- It is easy to see

$$||\bar{y}_N||_p = (S_N^p(x) + (zz^*)^{p/2})^{1/p}$$

is not smaller than $||S_N(x)||_p$ and not larger than

$$||S_N(x)||_p + ||zz^*||_{p/2}^{1/2} = ||S_N(x)||_p + ||z^*z||_{p/2}^{1/2} = 2||S_N(x)||_p.$$

- Thus this proves the Burkholder-Gundy inequality in the range $p \geq 2$, with upper and lower constants of order $O(p)$ as $p \to \infty$ which is optimal.

Classical case

- The notion of strong differential subordination was also introduced by Burkholder in 1994.
- Let $f = (f_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$ be two adapted sequences of integrable random variables with the corresponding differences $df = (df_n)_{n\geq 0}$, $dg = (dg_n)_{n\geq 0}$. We say that g is strongly differentially subordinate to f if the following conditions are satisfied:

(DS) for any $n \geq 0$ we have $|dg_n| \leq |df_n|$;
(CDS) for any $n \geq 1$ we have $|\mathbb{E}_{n-1}(dg_n)| \leq |\mathbb{E}_{n-1}(df_n)|$.

Remark If f and g are martingales, then the second condition (2) is automatically satisfied.

Theorem (Burkholder, 1994)

Suppose that f is a nonnegative submartingale and g is strongly differentially subordinate to f. Then

$$||\sup_{n\geq 0}|g_n|||_{1,\infty} \leq 3||f||_1; \tag{6}$$

and

$$||g||_p \leq (p^* - 1)||f||_p, \quad 1 < p < \infty, \tag{7}$$

where $p^* = \max\{2p, p/(p-1)\}$. Here the constants are both sharp.

D. L. Burkholder, *Strong differential subordination and stochastic integration*, Ann. Probab. **22** (1994), 995–1025.

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Q: noncommutative variants of (6) and (7)?

NC case

I. Proper definition of NC SDS

Let x, y be self-adjoint adapted sequences of operators.

- NC (DS) condition. Two alternative candidates:

  (DS) for any $n \geq 0$ and any projection $R \in \mathcal{M}_{n-1}$,

$$Rdy_nRdy_nR \leq Rdx_nRdx_nR;$$

  (WDS) for any $n \geq 0$,

$$dy_n^2 \leq dx_n^2.$$

  Why consider these two different candidates? Similar to the NC martingale case (recall the example).

- NC (CDS) condition.

(CDS) For any $n \geq 1$,

$$-|\mathcal{E}_{n-1}(dx_n)| \leq \mathcal{E}_{n-1}(dy_n) \leq |\mathcal{E}_{n-1}(dx_n)|.$$

Remark

(i) Of course, in the classical case,

$$-|\mathbb{E}_{n-1}(dx_n)| \leq \mathbb{E}_{n-1}(dy_n) \leq |\mathbb{E}_{n-1}(dx_n)|$$

is equivalent to

$$|\mathbb{E}_{n-1}(dy_n)| \leq |\mathbb{E}_{n-1}(dx_n)|.$$

(ii) However, in the NC case, the above condition is weaker (but sufficient!) than

$$|\mathcal{E}_{n-1}(dy_n)| \leq |\mathcal{E}_{n-1}(dx_n)|.$$

Definition (Jiao, Osękowski and Wu, AOP, 2019)

Let x, y be self-adjoint adapted sequences of operators.
(i) y is strongly differentially subordinate to x if (DS)+(CDS):
(DS) for any $n \geq 0$ and any projection $R \in \mathcal{M}_{n-1}$,

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(ii) y is "weakly" strongly differentially subordinate to x if (WDS)+(CDS):
(WDS) for any $n \geq 0$,

$$dy_n^2 \leq dx_n^2.$$

(CDS) For any $n \geq 1$,

$$-|\mathcal{E}_{n-1}(dx_n)| \leq \mathcal{E}_{n-1}(dy_n) \leq |\mathcal{E}_{n-1}(dx_n)|.$$

Theorem (Jiao, Osękowski and Wu, AOP, 2019)

Let x be a nonnegative submartingale and y be a self-adjoint adapted sequence of operators. Suppose that y is strongly differentially subordinate to x (i.e. (DS)+(CDS) are satisfied). Then there exists a projection q satisfying

$$-q \leq qy_nq \leq q, \quad \text{for all } n,$$

and such that

$$\tau(I-q) \leq 327||x||_1.$$

II. Main Results: weak type inequality

Theorem (Jiao, Osękowski and Wu, AOP, 2019)

Let x be a nonnegative submartingale and y be a self-adjoint adapted sequence of operators. Suppose that y is strongly differentially subordinate to x (i.e. (DS)+(CDS) are satisfied). Then there exists a projection q satisfying

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$$I - q \sim \{ \sup_{\{n \geq 0\}} |g_n| > 1\};$$

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Remark (i)

$$I - q \sim \{\sup_{\{n\geq 0\}} |g_n| > 1\};$$

(ii) The proof depends on a novel asymmetric Gundy's decomposition and two-sided Cuculescu's projections for submartingales.

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Theorem (Jiao, Osękowski and Wu, AOP, 2019)

Suppose that x and y are as above and (DS)+(CDS) are satisfied. Then for $1 < p < \infty$, we have

$$\|y\|_p \leq c_p ||x||_p;$$

Moreover, if $p \geq 2$, then the inequality above holds true under the weaker assumption that y is 'weakly' strongly differently subordinate to x (i.e., (WDS)+(CDS) are satisfied).

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Remark The constant C_p is of orders $O((p-1)^{-1})$ as $p \to 1_+$ and $O(p^4)$ as $p \to \infty$.

Idea of proof

(i) Doob-Meryer decomposition:

submartingale=martingale+predictable process.

(ii) Case $1 < p < 2$:

NC Bellman function method $\to$ predictable process;

Gundy's decomposition+ K-functional $\to$ martingale part.

(iii) Case $2 < p < \infty$:

NC good-λ approach $\to$ predictable process;

An estimate for submartingale differences $\to$ martingale part.

Square Functions of Differential Subordinate Martingales

Classical square functions and related results

- Let $f = (f_n)_{n\geq 1}$ be a classical martingale. Define

$$S(f) = \Big(\sum_{k=1}^{\infty} |df_k|^2\Big)^{1/2}.$$

- (Burkholder 1971)

$$\begin{aligned} \|S(f)\|_{1,\infty} &\leq c\|f\|_1; \\ \|S(f)\|_p &\leq c_p\|f\|_p, \quad 1 < p < \infty. \end{aligned} \tag{8}$$

- If martingale g is differentially subordinate to f, then

$$\begin{aligned} \|S(g)\|_{1,\infty} \leq \|S(f)\|_{1,\infty} &\leq c\|f\|_1; \\ \|S(g)\|_p \leq \|S(f)\|_p &\leq c_p\|f\|_p, \quad 1 < p < \infty. \end{aligned}$$

Q: NC case is also trivial or not? The answer is no!

NC square functions

- Let $x = (x_n)_{n\geq 1}$ be a noncommutative martingale. Define

$$S_c(x) = \Big(\sum_{k=1}^{\infty} |d_k x|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |(d_k x)^*|^2\Big)^{1/2}.$$

- (Pisier-Xu 1997)There are martingales a, b such that $x = a + b$ and

$$\|S_c(a)\|_p + \|S_r(b)\|_p \leq c_p \|x\|_p, \quad 1 < p < 2.$$

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Q: Similar results for differentially subordinate martingales.

Main results

Theorem (Jiao, Randrianantoanina, Wu and Zhou, CMP, 2019)

Let x, y be two self-adjoint martingales such that y is weakly differentially subordinate to x. Then there are two martingales a and b such that $y = a + b$. Moreover, we have

$$\|S_c(a)\|_{L_{1,\infty}(\mathcal{M})} + \|S_r(b)\|_{L_{1,\infty}(\mathcal{M})} \leq c\|x\|_1,$$

and

$$\|S_c(a)\|_p + \|S_r(b)\|_p \leq c_p\|x\|_p, \quad 1 < p < 2,$$

where c_p is of order $(p-1)^{-1}$ as $p \to 1$. This order is optimal.

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Remark:

- $\big\|y\big\|_{H_p} \leq c_p\big\|x\big\|_{H_p}$ with $c_p = O((p-1)^{-1})$ when $p \to 1$.

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Remark:

- $\big\|y\big\|_{H_p} \leq c_p\big\|x\big\|_{H_p}$ with $c_p = O((p-1)^{-1})$ when $p \to 1$.
- Similar results for conditional square functions.

Idea for weak type inequality

- Triangular truncation $\to y = a + b$.
- New asymmetric Gundy's decomposition to estimate a, b separately.

Idea for strong type inequality

- Note that $\|z\|_p^p = \int_0^\infty pt^{p-1}\tau(\chi_{[t,\infty)}(|z|))dt$ for $z \in L_p(\mathcal{M})$.
- Key lemma:

$$\tau\Big(\chi_{[2^k,\infty)}(S_c(a))\Big) \lesssim_{c_{abs}} \cdot 2^{-2k}\|e_{k,N}x_N e_{k,N}\|_2^2 + \cdot 2^{-k}\tau((I - e_{k,N})|x_N|).$$

  Similar holds true for $\tau\Big(\chi_{[2^k,\infty)}(S_r(b))\Big)$.

- Multiply both sides with 2^k and take the sum over k.

Other Related Problems

- Let D be an open connected subset of $\mathbb{R}^n$ and H be a real or complex Hilbert space with norm $|\cdot|$. Suppose that u, $v : D \to H$ are harmonic. We say that v is differentially subordinate to u if for all $x \in D$,
$$|\nabla v(x)| \leq |\nabla u(x)|.$$
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Theorem (Burkholder, 1989)

If v is differentially subordinate to u, then

$$\begin{aligned} &\mu(|v| \geq 1) \leq 2||u||_1; \\ &||v||_p \leq (p^* - 1)||u||_p, \quad 1 < p < \infty, \end{aligned} \tag{9}$$

where $p^* = \max\{p, p/(p-1)\}$.

- Question 1: Can we give a proper definition of NC differential subordination for harmonic functions?

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- If so, can we extend this result to the noncommutative setting?

- As the classical case,

  Question 2: Noncommutative differential subordinations of continuous-time martingales?

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- Question 3: Does there exist a constant C so that if y is a self-adjoint martingale that is weakly differentially subordinate to another self-adjoint martingale $x \in H_1$, then

$$\|y\|_{H_1} \leq C\|x\|_{H_1}?$$

Thank You !